Part II: Differential Topology

Answer all questions. In your proofs, you may use any major theorem, except the fact you are trying to prove (or a variant of it). State clearly what theorems you use. Good luck.

Question 1

Let M be a smooth manifold and V, W smooth vector fields.

a) Prove that $L_V W = [V, W]$.

b) Let V, W be the vector fields on $\mathbb{R}^2$ given by

$$V = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \qquad W = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

Find their flows.

c) Do the flows V, W commute?

d) If they do commute, find the coordinate function centered at $(1, 0)$ with V, W as the coordinate vector fields.

Question 2

Let $F : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ be given by

$$F(x) = \frac{x}{||x||^2}$$

where $||x||$ is the euclidean norm.

a) Find the differential dF_x and show that with respect to it is a composition of a reflection in the plane perpendicular to x followed by a scaling by a factor of $1/||x||^2$.

b) If ω is the euclidean volume form, find $F^*\omega$.

Question 3

a) Let $F : G \to H$ be a Lie group homomorphism and let $F_* : G \to H$ be the map between the associated Lie algebras of left-invariant vector fields defined by letting $(F_*(X))_e = dF_e(X_e)$.
Show that F_* is a Lie algebra homomorphism.

b) State the equivariant rank theorem.

c) Prove that $O(n)$ the group of orthogonal linear maps is a manifold and find its dimension.

Question 4

a) Give the definition of the integral of an n-form on an oriented n-manifold and show it is well-defined.

b) State and prove Stokes Theorem.

Question 5

a) State the Cartan Magic Formula.

b) Let M be a smooth manifold and $i_t : M \to M \times I$ be the map $i_t(x) = (x, t)$.
Show that $i_0^*, i_1^* : \Omega(M \times I) \to \Omega(M)$ are cochain homotopic, i.e., there exists a collection of linear maps h