

You should attempt the problems. Partial credit will be given for serious efforts.

- (1) Let S be a closed non-orientable surface of genus g .
 - (a) What is $H_i(S; \mathbb{Z}_2)$? (answer only)
 - (b) Find out the maximal number of disjoint orientation reversing simple closed curves in S . (Justify your answer)
- (2) Let X be a path-connected space and \tilde{X} a universal covering space of X . Prove that if \tilde{X} is compact, then $\pi_1(X)$ is a finite group.
- (3) Let M be a compact, connected, orientable n -manifold, where n is odd. (You may assume, if you like, that M is triangulated.)
 - (a) Show that if $\partial M = \emptyset$, then $\chi(M) = 0$.
 - (b) Show that if $\partial M \neq \emptyset$, then $\chi(M) = \frac{1}{2}\chi(\partial M)$.
- (4) Let M be a closed 3-manifold. Suppose M is a homology sphere, i.e., M has the same \mathbb{Z} -coefficient homology groups as S^3 , in other words, $H_n(M; \mathbb{Z}) = H_n(S^3; \mathbb{Z})$ for all n . Let k be a knot in M (i.e., k is a closed 1-dimensional submanifold of M , in other words, k is an embedded closed curve S^1 in M). Compute $H_n(M - k; \mathbb{Z})$ for all n , where $M - k$ is the complement of k .

GT Qual 2011 Part II
Show All Relevant Work!

1) The image of the map $X : \mathbf{R}^2 \rightarrow \mathbf{R}^3$ given by

$$X(\theta; \phi) = ((2 + \cos(\phi)) \cos(\theta); (2 + \cos(\phi)) \sin(\theta); \sin(\phi))$$

is the torus obtained by revolving the circle $(y - 2)^2 + z^2 = 1$ in the yz plane about the z axis. Consider the map $F : \mathbf{R}^3 \rightarrow \mathbf{R}^2$ given by $F(x; y; z) = (x; z)$ and let $f = (F$ restricted to the torus).

a) Compute the Jacobian of the map $f \circ X$: (Note that the map X descends to an embedding of $S^1 \times S^1$ into \mathbf{R}^3 but we don't need to obsess over the details of this.)

b) Find all regular values of f .

c) Find all level sets of f that are *not* smooth manifolds (closed embedded sub-manifolds).

2a) Write down the deRham homomorphism for a smooth manifold M ; explain briefly why this definition is independent of the (two) choices made.

b) State the deRham Theorem for a smooth manifold M .

c) A crucial step in the proof of the deRham Theorem is: If M is covered by 2 open sets U and V , both of which and their intersection satisfy the deRham theorem, then $M = U \cup V$ satisfies the deRham theorem. Briefly explain how this crucial step is proven.

3a) If ω is a differential form, then must it be true that $d\omega = 0$? If yes, then explain your reasoning. If no, then provide a counterexample.

b) If ω and η are closed differential forms, prove that $\omega \wedge \eta$ is closed.

c) If, in addition (i.e., continue to assume that ω is closed), η is exact, prove that $\omega \wedge \eta$ is exact.

4) The Chern-Simons form for a hyperbolic 3-manifold with the orthonormal framing $(E_1; E_2; E_3)$ is the 3-form

$$Q = \left(\frac{1}{8}\right) (\omega_{12} \wedge \omega_{13} \wedge \omega_{23} - \omega_{12} \wedge \omega_1 \wedge \omega_2 - \omega_{13} \wedge \omega_1 \wedge \omega_3 - \omega_{23} \wedge \omega_2 \wedge \omega_3)$$

where $(\omega_1; \omega_2; \omega_3)$ is the dual co-frame to $(E_1; E_2; E_3)$ (note that [Lee] uses θ , but here we use ω) and the ω_{ij} are the *connection* 1-forms. The connection 1-forms satisfy

$$d\omega_1 = \omega_{12} \wedge \omega_2 - \omega_{13} \wedge \omega_3$$

$$d\omega_2 = +\omega_{12} \wedge \omega_1 - \omega_{23} \wedge \omega_3$$

$$d\omega_3 = +\omega_{13} \wedge \omega_1 + \omega_{23} \wedge \omega_2$$

a) In $\mathbf{H}^3 = f(x; y; z) : z > 0$ with the Riemannian metric $g = \frac{1}{z^2} dx^2 + \frac{1}{z^2} dy^2 + dz^2$, orthonormalize the framing $(\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z})$:

b) Compute the associated dual co-frame $(\omega_1; \omega_2; \omega_3)$:

c) For this orthonormal framing (and dual co-frame), in $(\mathbf{H}^3; g)$, compute the Chern-