

Analysis Qualifying Exam -----Spring 2011

Please answer all 6 problems from each section and show your work. Each problem is worth 30 points.

SECTION 1: REAL ANALYSIS

In your proofs, you may use any major theorem, except the fact you are trying to prove, or a mere variant of it. State clearly what theorems you use. Good luck.

1 Let $f_n : X \rightarrow \mathbb{R}$ be (X, μ) measurable functions such that $0 \leq f_n \leq 1$ and μ is a finite measure. Prove that if

$$\lim_n \int_X f_n d\mu = \mu(X)$$

then $f_n \rightarrow 1$ a.e.

2 Let f be Lebesgue integrable on $(0, a)$ and $g(x) = \int_x^a t^{-1} f(t) dt$.
Prove that g is Lebesgue integrable on $(0, a)$ and $\int_0^a g(x) dx = \int_0^a f(x) dx$.

3 Let μ_n be finite measure on (X, M) with $\mu_n \ll \mu$. Define $\nu = \mu + \sum \mu_n$ and $f = \frac{d\mu}{d\nu}$.
Prove that $\int f d\mu = \int f d\nu$.

4 Let $s \in \mathbb{C}$ and consider

$$\sum_{n=1}^{\infty} n^{-s}.$$

- a) Prove that the sum converges absolutely and uniformly in the set $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > 1\}$ for any $\sigma > 1$.
- b) Assume the result in part a). Then the sum defines an analytic function on $U := \{s \in \mathbb{C} \mid \operatorname{Re}(s) > 1\}$