

Real Analysis Qual

June 10, 2016

Problem 1. Let \sim be the equivalence relation on the interval $[0, 1]$ given by $x \sim y \iff x - y \in \mathbb{Q}$. Choose one element from each equivalence class (using the Axiom of Choice). Let $A \subset [0, 1]$ denote the set of these chosen elements. For a given set $B \subset \mathbb{R}$, define $B + x = \{y + x \mid y \in B\}$.

- (a) Show that the sets $A + q$, for $q \in \mathbb{Q} \cap [-1, 1]$, are disjoint.
- (b) Show that $[0, 1] \subset \bigcup_{q \in \mathbb{Q} \cap [-1, 1]} (A + q) \subset [-1, 2]$.
- (c) Show that A

(c) Prove Fatou's Lemma, which states that

$$\liminf_n \int f_n \leq \int \liminf_n f_n$$

for any sequence (f_n) of non-negative, measurable functions. To get started, let $g_n = \inf_{i \geq n} f_i$. Observe that

$E_1, \dots, E_n \subset A$ and $c_1, \dots, c_n \in \mathbb{R}_0^+$. Define $\nu : A \rightarrow [0, \infty]$ by $\nu(X) < \infty$. Fix

$$\nu(A) = \sum_{i=1}^n c_i \mu(A \cap E_i).$$

- (a) Show that ν is a measure.
- (b) Show that ν is absolutely continuous with respect to μ .

Complex Analysis Qualifying Exam – Spring 2016

Please answer the following problems. Explain your argument carefully – if you refer to a well-known theorem from class, please state the theorem precisely and explain why it applies.

Notation: D = open unit disk, $\mathbb{C}^* = \mathbb{C} - \{0\}$, H = upper half plane.

- 1) Find a holomorphic function f on \mathbb{C}^* such that
 - f is a pointwise limit of polynomial functions, but
 - f is not a uniform limit of polynomial functions (that is, there is no sequence of polynomials that converges to f uniformly on compact subsets of \mathbb{C}^*).

Prove both assertions for your choice of f .

- 2) Find a biholomorphism between H and the region

$$U = \{z \in \mathbb{C} \mid |z - 1| < 1, |z - i| < 1\}.$$

It is enough to write down explicitly functions whose composition yields a biholomorphism from H to U or from U to H .

- 3) Let $U \subset \mathbb{C}$ be a simply connected region. For any point $a \in U$, the Green function of U

where $\zeta(z)$ is the Weierstrass ζ -function

$$\zeta(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right).$$

Justify carefully any techniques you use.