Intransitivity in the Small and in the Large

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Abstract

We propose a regret-based model that allows the separation of attitudes towards transitivity on triples of random variables that are closethat are far apart. This enables a theoretical reinterpretation of evidence related to intransitive behavior in the laboratory. When viewed through this paper's analysis, the experimental evidence need not imply intransitive behavior for large risky decisions such as investment choices and insurance.

Keywords: Intransitivity, regret, local preferences, preference reversal

Introduction

precedes C then A precedes C seems almost obvious. Yet we know that not

put enough zeroes on the ends of the payoffs ..., you will observe subjective expected utility behavior.” In our view, this argument is especially relevant for violations of transitivity because such violations may be due to insufficient consideration by decision makers. That is, cycles may be observed with respect to small gambles, but as decision makers will pay more attention and exert more effort when making big financial decisions, cycles are less likely to happen. If this is the case, then unlike violations of the independence axiom and its alternatives, where experiments may reveal insight into real-world decision making, observing violations of transitivity in experiments does not necessarily indicate such behavior in the large. Our aim is to provide a formal

intransitivity in one part of the domain implies intransitivity everywhere. A more general regret model, which permits a decoupling of attitudes towards transitivity in the small and transitivity in the large, is considered in Section 3. All proofs are in Appendix A while Appendix B contains several examples.

Linear Regret

Consider a set $\mathcal{L}$ of finite-valued random variables X of the form X = $(x_1, s_1; \ldots; x_n, s_n)$ where the outcomes are monetary payoffs (which may be positive or negative).[4] The events $s_1, \ldots, s_n$ partition the sure event and the probability of s_i is s variables $X = (x_1, s_1; \ldots; x_n, s_n)$ and $Y = (y_1, s_1; \ldots; y_n, s_n)$ over the same set of events,

$$X \succeq Y \quad \text{if and only if} \quad \sum_i \Pr[s_i]\, \psi(x_i, y_i) \,\square\, 0 \qquad (1)$$

where ψ is a *regret function* which is continuous and for all x and y,

(i) $\psi(x, y) = \ \psi(y, x)$,

(ii) ψ is increasing in its first and decreasing in its second argument.

The function ψ represents the feelings of the decision maker when [(;)-167.12-0.924067(i)-0.765802(

non-expected utility choice.[6] For clarity, we refer to preferences as defined in eqs. (1) or (2) as *universal regret*, as later we define local regret.

Assume linear regret. If for some x_1, x_2, x_3 and s_1, s_2, s_3 such that $Pr[s_1] = Pr[s_2] = Pr[s_3] = \frac{1}{3}$, $(x_1, s_1; x_2, s_2; x_3, s_3)$ $(x_3, s_1; x_1, s_2$

The proof of Theorem 1(i) makes specific predictions that can be checked experimentally. A violation of transitivity implies a cycle as in eq. (3). Then for every y there is a sufficiently small $\varepsilon > 0$ such that for $s_0, \ldots, s_3$ where $\Pr[s_0] = 1 - \varepsilon$ and $\Pr[s_1] = \Pr[s_2] = \Pr[s_3] = \frac{\varepsilon}{3}$,

$$(y, s_0; x_1, s_1; x_2, s_2; x_3, s_3) \qquad (y, s_0; x_3, s_1; x_1, s_2; x_2, s_3)$$

$$(y, s_0; x_2, s_1; x_3, s_2; x_1, s_3) \qquad (y, s_0; x_1, s_1; x_2, s_2; x_3, s_3)$$

Theorem 1 strongly depends on the assumption that regret is linear in probabilities, but it does not hold for non-linear models of regret. Example 1 in Appendix B provides a regret relation that is transitive in the large, yet violates transitivity in the small. Once regret is not linear in probabilities, the opposite is also possible. Example 2 in Appendix B presents a non-linear model of regret which is expected utility (and therefore transitive) in every small neighborhood, yet has intransitive cycles in the large.

Although non-linear regret permits a separation between attitudes towards transitivity in the small and in the large, it nevertheless imposes some strict restrictions over preferences in small neighborhoods. We analyze such preferences in the next section.

Local preferences and regret

To facilitate a distinction between intransitive cycles where random variables are far away from each other and cycles where random variables are all in small neighborhoods, define preferences to be *locally regret-based* if they can be represented as in eq. (2) above in a neighborhood around each random variable W, albeit possibly with different functions ψ and V. Formally, a binary relation is locally regret-based if for every W there is $\varepsilon > 0$ such that for all $X, Y \in \mathcal{B}(W, \varepsilon)$,

$$X \succeq Y \quad \text{if and only if} \quad V_W(\Psi_W(X, Y)) \geq 0$$

As we show below, local regret does not imply universal regret, yet it does impose restrictions on ψ_W and V_W across different values of W (Theorem 2). First, all the local-regret functions ψ_W can be taken to be the same. Second, any two local-regret functionals V_W, $V_{W'}$ with domains $\mathcal{R}_W, \mathcal{R}_{W'}$ are

Proposition *If preferences are locally regret-based, then* $\mathbf{X}^1 \quad \mathbf{Y}^1$ *if and only if* $\mathbf{X}^2 \quad \mathbf{Y}^2$, *where* $\mathbf{X}^j, \mathbf{Y}^j$ *are defined above.*

This proposition is related to Savage's sure-thing principle. The difference is that in Proposition 1 the common parts of $\mathbf{X}^j$ and $\mathbf{Y}^j$ are "large" while these is no such restriction in the sure-thing principle.

If each local-regret functional is linear in probabilities, then we have a stronger result than Theorem 2.

Proposition *If preferences are locally regret-based and each local-regret functional* $\mathbf{V}_W$ *is linear in probabilities, then local regrets are identically linearly evaluated. That is, each local regret is the expected value of a common (up to positive multiplication) local-regret function* ψ *for all* $\mathbf{W}$.

In the proof of Theorem 2(i), the ordinal equivalence of the ψ_W functions is obtained by adjusting the regret functionals. The adjusted regret functional will in general be non-linear, even if the initial regret functional is linear. Thus, Proposition 2 does not follow from Theorem 2(i).

Remark If preferences are locally regret-based then Theorem 2 implies that either there are intransitive cycles in every neighborhood or there is no intransitive cycle in any neighborhood. Intransitivity in some but not all neighborhoods are possible when preferences do not satisfy local regret.

Remark Our distinction between preferences in the small and in the large should not be confused with Machina's [17] model of Fréchet differentiable representations, where preferences violate the independence axiom while converging at each point to expected utility. Intransitive regret models of the type discussed in this paper do not permit a representation function (which necessarily implies transitivity), hence are orthogonal to Machina's analysis.

4 Discussion

Example 1 in Appendix B shows that violations of transitivity when random variables are close to each other do not imply the existence of intransitive cycles when random variables are far apart from each other. As experiments are done with “small” random variables, it is questionable to what extent one may deduce from these experiments that individuals violate transitivity in “big” decisions like financial investments, real-estate transactions, or retirement planning.

But isn’t this true for all experimental results? For example, when real payments are involved, experiments regarding the Allais paradox (Allais [1]; see also MacCrimmon and Larsson [16], Kahneman and Tversky [12], and Starmer [22]) are conducted, for obvious reasons, with small amounts of money. Will an argument similar to the one made in the paper lead to the conclusion that we cannot learn from these experiments that the Allais paradox really exists?

There is however an important difference between experiments on tranyet such preferences persist even after such modifications (see for example

$$\frac{\psi(x_1, x_3) + \psi(x_2, x_1) + \psi(x_3, x_2)}{3} > 0$$

Let $W = (w_1, t_1; \ldots; w_\ell, t_\ell) \in \mathcal{L}$. For any $m > \frac{1}{\varepsilon}$, let $s_1, \ldots, s_{3m}$ be pairwise disjoint with the probabilities $\frac{1}{3m}$

for any regret level r and integer $\ell \geq 2$. The first equality is true as (r, $\frac{1}{\ell}$; r, $\frac{1}{\ell}$; 0

$$
\begin{aligned}
&> V_i\left(\psi_i(x,y), \frac{1}{\square}; \psi_i(x,y), \frac{1}{\square}; 0, \frac{\square \quad 2}{\square}\right) \\
&= 0
\end{aligned}
$$

where we use the fact that ψ is skew symmetric, $\psi_i(x, y) < \psi_i(x, y)$, the monotonicity of V (

Let $\mathcal{R}_W$ be the set of regret lotteries generated by $\mathbf{X}, \mathbf{Y} \in \mathcal{B}(\mathbf{W}, \varepsilon)$ and $\mathcal{R}_{W'}$ be the set of regret lotteries generated by $\mathbf{X}\ , \mathbf{Y}\ \in \mathcal{B}(\mathbf{W}\ , \varepsilon)$. The set $\mathcal{R}_W \quad \mathcal{R}_{W'}$ is non-empty as $(0, 1)$ belongs to it. As $\mathcal{R}_W$ and $\mathcal{R}_{W'}$ are open sets so is $\mathcal{R}_W \quad \mathcal{R}_{W'}$. Therefore, we may take $\mathbf{R} \in \mathcal{R}_W \quad \mathcal{R}_{W'}$ such that $\mathbf{R} = (0, 1)$. Thus, there exist $\mathbf{X}, \mathbf{Y} \in \mathcal{B}(\mathbf{W}, \varepsilon)$, $\mathbf{X} = \mathbf{Y}$ and $\mathbf{X}\ , \mathbf{Y}\ \in \mathcal{B}(\mathbf{W}\ , \varepsilon)$, $\mathbf{X}\ = \mathbf{Y}$ such that $\mathbf{R} = \Psi(\mathbf{X}, \mathbf{Y}) = \Psi(\mathbf{X}\ , \mathbf{Y}\ )$. Without loss of generality we may write $\mathbf{X}, \mathbf{X}\ , \mathbf{W}$, and $\mathbf{W}$ on the same list of events $s_1, \ldots, s_n$. We can partition each s_i into two sub-events, $s_{i,\alpha}$ and $s_{i,}$

then R is locally generated in the neighborhood of each random variable on the line segment joining W and W .

Suppose that V_W and $V_{W'}$ are not concordant. In particular, V_W(R) > 0 and $V_{W'}$

Proof of Proposition : We first prove that if $X^1 \quad Y^1$, then $\psi_W(x_1, y_1) = \psi_W(x_2, y_2)$.

By the skew-symmetry of the functions ψ,

$$X^1 \quad Y^1 \qquad V_W(\psi_W(x_1, y_1), \delta; \psi_W(x_2, y_2), \delta; 0, 1 \quad 2\delta) = 0$$
$$V_W(\quad \psi_W(y_1, x_1), \delta; \quad \psi_W(y_2, x_2), \delta; 0, 1 \quad 2\delta) = 0$$

Therefore, $\psi_W(x_1, y_1) \square \quad \psi_W(y_2, x_2)$ i $\psi_W(x_2, y_2) \square \quad \psi_W(y_1, x_1)$. However, $\psi_W(x_1, y_1) > \quad \psi_W(x_2, y_2)$ implies

$$\psi_W(x_1, y_1) = \psi_W(y_1, x_1) < \quad \psi_W(y_2, x_2) = \psi_W(x_2, y_2) =$$
$$V_W(\psi_W(y_1, x_1), \delta; \psi_W(y_2, x_2), \delta; 0, 1 \quad 2\delta) <$$
$$V_W(\psi_W(x_1, y_1), \delta; \psi_W(x_2, y_2), \delta; 0, 1 \quad 2\delta) = 0$$

A contradiction to $X^1 \quad Y^1$.

By Theorem 2(i), ψ_W is an increasing ordinal transformation of ψ_W. Therefore, $\psi_W(x_1, y_1) = \quad \psi_W(x_2, y_2)$ implies that $\psi_W(x_1, y_1) = \quad \psi_W(x_2, y_2)$ and thus $X^2 \quad Y^2$. □

Proof of Proposition : Let $W = (w_1, s_1; w_2, s_2; \ldots; w_n, s_n)$. Let ψ_W be a local-regret function at W. First, we show that the linearity of V_W implies that each ψ_W is unique up to positive multiples.

For two regret levels $r_1, r_2 > 0$, let $x, y > 0$ be monetary outcomes such that $r_1 = \psi_W(x, \quad x)$ and $r_2 = \psi_W(y, \quad y)$. Define

$$X = (x, s_{1,\varepsilon}; \quad y, s_{1,\varepsilon}; w_1, s_{1,1-\varepsilon \; -\varepsilon}; w_2, s_2; \ldots; w_n, s_n)$$
$$Y = (\quad x, s_{1,\varepsilon}; y, s_{1,\varepsilon}; w_1, s_{1,1-\varepsilon \; -\varepsilon}; w_2, s_2; \ldots; w_n, s_n)$$

where $\varepsilon_1, \varepsilon_2 > 0$, $\varepsilon_1 + \varepsilon_2 < 1$, $\varepsilon_2 / \varepsilon_1 = r_1 / r_2$, and $\Pr[s_{1,\varepsilon_\ell}] = \varepsilon_\ell$, $\ell = 1, 2$. Choose $\varepsilon_1, \varepsilon_2$

$$= \mathbf{r}_1 \boldsymbol{\varepsilon}_{1\cdot} \quad \mathbf{r}_2 \boldsymbol{\varepsilon}_2$$
$$= \mathbf{0}$$

Hence, $\mathbf{X} \quad \mathbf{Y}$.

Let $\hat{\boldsymbol{\psi}}_W$

that ψ_{j+1} is a positive multiple of ψ_j. Consequently, $\psi_{W'}$ is a positive multiple of ψ_W. □

Appendix B: Examples

Example INTRANSITIVE IN THE SMALL, TRANSITIVE IN THE LARGE

Let $\psi(x, y) = x \quad y$ be a regret function. For $X = (x_1, s_1; \ldots; x_n, s_n)$, $Y = (y_1, s_1; \ldots; y_n, s_n)$, $\Pr[s_i] = p_i$ define

$$V(\Psi(X, Y)) = \begin{cases} \sum_{i=1}^{n} p_i \psi(x_i, y_i) & \text{if} \quad \Psi(X, Y) \quad \geq \varepsilon \\ \alpha_{XYp} & \\ & \end{cases}$$

such that

$$V\left(cR_1 + (1 \quad c)R_2\right) = cV(R_1) + (1 \quad c)V(R_2)$$

Take two regret lotteries, R_1, R_2, with $R_1 > \varepsilon$ and $R_2 < \varepsilon$. Let c be su□ciently close to 1 so that $cR_1 + (1 \quad c)R_2 > \varepsilon$. Then

$$\begin{aligned} V\left(cR_1 + (1 \quad c)R_2\right) &= cE[R_1] + (1 \quad c)E[R_2] \\ &= cV(R_1) + (1 \quad c)V(R_2) \end{aligned}$$

as $E[R_1] = V(R_1)$ but $E[R_2] = V(R_2)$.

Example Transitive in the small, intransitive in the large

Let X, Y, and $\psi(x, y)$ be as in Example 1. Define

$$V(\Psi(X, Y)) = \begin{cases} \sum_{i=1}^{n} p_i \psi(x_i, y_i) & \text{if} \quad \Psi(X, Y) < \varepsilon \\ (1 \quad \beta_{XY}) \sum_{i=1}^{n} p_i \psi(x_i, y_i) + & \\ \quad \beta_{XY} & \end{cases}$$

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